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ASYMPTOTIC MOTIONS OF MECHANICAL SYSTEMS WITH NON-HOLONOMIC CONSTRAINTS*

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The motions of mechanical systems with non-holonomic constraints close to critical points of the potential are considered. The stability of the equilibrium positions was first treated by Whittaker /1/. A theorem is given which includes earlier results /2/ as a special case, and which enables asymptotic motions to be found for new classes of potentials. Sufficient conditions are found for the equilibrium to be unstable when not all the frequencies of small oscillations vanish. Similar studies were made in /3-6/ for systems without constraints.

The hypothesis can be advanced that a critical point of the potential energy is an unstable equilibrium of a mechanical system with non-holonomic constraints (linear in the velocity) when zero is not a minimum of the function V^*.

Here, the origin is the equilibrium position in question, and the asterisk denotes contraction of the potential energy V to the subspace, orthogonal to all the constraints at zero.

This hypothesis is proved below for the case when the MacLaurin expansion of V^* is $V^* = V_2^* + V_k^* + V_{k+1}^* + \cdots$, where $V_2^* + V_k^*$ can take negative values infinitesimally close to zero (V_j^* is a homogeneous form of degree j).

This situation when $V_2^* \geqslant 0$ and $V_k^* \geqslant 0$ is not considered. Also, to determine the absence of a minimum, higher powers must be taken into account.

*1. **The rigorous statement of the problem, and the result.*** We consider a mechanical system with configuration space L, which can be regarded as the standard R^n, since all our constructions are performed in an infinitesimally small neighbourhood of zero. Let the generalized coordinates be $\xi = (\xi^1, \ldots, \xi^n)^T \in L$. The Lagrangian of the system can be written as $K(\xi, \xi^{\cdot}) - V(\xi)$, where K is the kinetic energy, quadratic in the velocity, and V is the

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potential energy, which is assumed to be an analytic function of its arguments. In addition, we impose on the system m constraints, linear in the velocities $(0 \leqslant m < n)$:

$$(a_j(\xi), \xi^\cdot) = 0, \quad j = 1, 2, \ldots, m \tag{1.1}$$

In general, the constraints are not integrable, and the vectors $a_j(\xi)$ are assumed to be linearly independent at all points $\xi \in L$. The parentheses denote the scalar product in the sense of the metric given by the kinetic energy.

We introduce the $n \times m$ matrix $A(\xi)$, composed of the $a_j(\xi)$. The dynamic behaviour of the system is then described by the Lagrange equations with the bounds

$$\frac{d}{dt}\frac{\partial K}{\partial \xi^\cdot} - \frac{\partial K}{\partial \xi} = -\frac{\partial V}{\partial \xi} + A(\xi)\lambda, \quad A^T(\xi)\xi^\cdot = 0 \tag{1.2}$$

Here, λ is a column of m arbitrary coefficients. We assume that the critical point of V is the same as the origin $0 \in L$. In the neigbourhood of this equilibrium position we can introduce normal coordinates, in which

$$K = \frac{1}{2}\sum_{i=1}^{n}(\xi^{\cdot i})^2 + \frac{1}{2}\sum_{i,j=1}^{n}k_{ij}(\xi)\xi^{\cdot i}\xi^{\cdot j}$$

We denote by $L_A \subset L$ the m-dimensional subspace generated by the vectors $a_j(0)$, and by $L_B \subset L$ its orthogonal complement (after introducing the normal coordinates, L becomes a Euclidean space).

Theorem. If the bound of the potential V in the subspace L_B has the form $V^* = V_2^* + V_k^* + V_{k+1}^* + \ldots$, where $0 \in L_B$ is not a local minimum of the function $V_2^* + V_k^*$, then this equilibrium position of system (1.2) is unstable.

Note that, as distinct from the results of /2/, it is not required here that the frequencies of all the small oscillations be zero.

The proof is based on the following lemma, which is of independent interest.

Lemma. If the hypotheses of the theorem hold and $V_2^* \geqslant 0$ everywhere in L_B, then the following formal solution of system (1.2) exists:

$$\xi^i(t) = \sum_{j=1}^{\infty}\xi_j^i(\ln t)\,t^{-j\mu}, \quad \mu = \frac{2}{k-2}$$

where $\xi_j^i(\tau)$ is a polynomial of finite degree in its argument $\tau = \ln t$.

The lemma is proved below. Let us show how the theorem is obtained from it. By the result of /7/, the existence of the above formal solution implies that system (1.2) has a real motion, for which the formal series $\xi(t)$ is the asymptotic expansion. Using the arguments of /8/ about time reversibility, we see that the equilibrium position is unstable. If the form V_2^* can take negative values in the neighbourhood of zero, we can prove the instability by Lyapunov's methods /9/.

The structure of the proof is as follows: we choose suitable coordinates and eliminate the multipliers λ, construct the zero approximation, then the first approximation, and the general form of the formal solution, and the conditions for solvability that arise at the N-th step

2. *Choice of coordinate system and elimination of Lagrange multipliers.* We eliminate the cofficients λ from the first n Eqs.(1.2), by differentiating the last m equations with respect to time and substituting into them the expression for $\xi^{\cdot\cdot}$. We obtain a system of n equations in the n unknown functions $\xi^i(t)$:

$$\xi^{\cdot\cdot *} = -V' + A(A^TA)^{-1}A^TV' + \Gamma(\xi, \xi^\cdot) + O(\xi)V'(\xi) \tag{2.1}$$

Here, V' is the gradient of V, $\Gamma(\xi, \xi^\cdot)$ is a vector function, quadratic in the velocities, $O(\xi)$ is a linear operator, $O(0) = 0$, and $A = A(0)$ is a constant matrix. For the asymptotic solution $\xi(t) \to 0$ and $\xi^\cdot(t) \to 0$ as $t \to \infty$, so that the eliminated equations of constraints (1.1) hold.

Applying the splitting lemma to the function V, we can introduce new coordinates (denoted as before by ξ^i), in which

$$V(\xi) = \frac{1}{2}\sum_{i=1}^{s}(\xi^i)^2 + V_3(\xi^{s+1}, \ldots, \xi^n) + \cdots$$

where the terms indicated by the dots have a degree of homogeneity greater than three and depend only on the coordinates $\xi^{s+1}, \ldots, \xi^n$.

We shall in future denote the coordinates $\xi^1, \ldots, \xi^s$ by $x^1, \ldots, x^s$ and the subspace generated by them by L_X.

Assume that we have the condition $L_A \cap L_X = \{0\}$. Otherwise, we should have to separate in L_A the subspace $L_Q = L_A \cap L_X$, and introduce into it a new basis, then make all the arguments below for the subspace orthogonal to L_Q. This makes the details of the proof much more complicated without affecting the general course of the arguments. To give the full picture, we note below the explicit form of the series that arise for the coordinates of subspace $L_A \cap L_X$ in the case when it is not empty.

We divide into three groups y, p, z, the variables that remain after separating L_X. Let L_Y, L_P, L_Z be the subspaces stretched over these groups. Variables p and z generate a subspace $L_P \oplus L_Z$, which is the intersection of $L_X{}^{\perp}$ with L_B. In other words, it is the subspace consisting of vectors orthogonal to all the $a_j(0)$ which do not participate in the quadratic terms of V^*. The subspace L_Y is defined as the orthogonal complement to $L_X \oplus L_P \oplus L_Z$. By our assumptions, $\dim L_Y = m$. By the hypothesis of the lemma, the contraction of V_k onto L_B does not have a local minimum at zero. Consequently, in any neighbourhood of zero in $L_P \oplus L_Z$ there is a point (p, z) for which $V_k{}^*(p, z) < 0$. Thus $V_k{}^*$ reaches its minimum on the unit sphere of $L_P \oplus L_Z$ and is negative at this point. We take the z axis in this extremal direction. We denote the remaining coordinates by p^i, choosing them in such a way that an orthonormalized basis is formed by the set x^i, y^i, p^i, z. We retain for it the notation ξ^i.

Henceforth $V(x, y, p, z)$ denotes $V(\xi)$, where $\xi^i = x^i$, $1 \leqslant i \leqslant s$; $\xi^i = y^{i-s}$, $s + 1 \leqslant i \leqslant s + m$; $\xi^i = p^{i-s-m}$, $s + m + 1 \leqslant i \leqslant n - 1$; $\xi^n = z$. Hence $V_k{}^*(p, z) = V(0, 0, p, z)$;

$$L = L_X \oplus L_Y \oplus L_P \oplus L_Z = L_A + L_B$$

$$\dim L = n,\ \dim L_X = s,\ \dim L_Y = m,\ \dim L_P = r = n - s - m - 1,$$

$$\dim L_Z = 1,\ \dim L_A = m,\ \dim L_B = n - m$$

Notice that only the subsets L_A stretched over the vectors $a_j(0)$, and not the vectors themselves, participate in (2.1). It is convenient to assume that the $a_j(0)$ are orthonormalized, and that the $(s + i)$-th coordinate is non-zero only in $a_i(0)$, i.e., the $n \times m$ matrix has the form $A^T = \| a\ b\ 0\ 0 \|$, where a is an $s \times m$ matrix, b is an $m \times m$ diagonal matrix, and the last two zeros denote $r \times m$ and $1 \times m$ blocks respectively. Note that, in view of the choice of subspace L_Y, all the elements b_{ii} are non-zero.

The orthonormalization condition for the $a_j(0)$ implies that $A^T A = E_m$ is the $m \times m$ unit matrix, so that (2.1) can be rewritten as

$$\xi^{\cdot\cdot} = TV'(\xi) + \Gamma(\xi, \xi^{\cdot}) + O(\xi)V'(\xi), \quad T = AA^T - E_n \tag{2.2}$$

where T is the operator of projection onto subspace L_B, taken with the opposite sign. The following assertions are important later.

Assertion 1. The matrix $aa^T - E_m$ is not degenerate.

The proof is obtained by using Hadamard's inequality.

Assertion 2. The $(s + i)$-th row of matrix T, $1 \leqslant i \leqslant m$, is a linear combination of the first s rows with weights $a_{i1}/b_{ii}, \ldots, a_{is}/b_{ii}$.

3. *The zero and first approximations.* We consider the simplified system

$$p^{\cdot\cdot} = -\frac{\partial V}{\partial p}(0, 0, p, z), \quad z^{\cdot\cdot} = -\frac{\partial V}{\partial z}(0, 0, p, z) \tag{3.1}$$

It has the solution $p_0(t) \equiv 0$, $z_0(t) = z_0 t^{-\mu}$, where $\mu = 2/(k - 2)$, and z_0 is a constant. The use of this solution was proposed in /8/ and is a consequence of taking the z axis in the direction of steepest descent of the function $V_k{}^*$. It was shown in /8/ that

$$V_k{}^*(p, z) = w(z)^k + \sum_{i,j=1}^{r} f_0(p, z)\, p^i p^j,$$

while z_0 is found from the relation $z_0\mu(\mu + 1) = -kw(z_0)^{k-1}$, which is always solvable by virtue of the condition that $V_k{}^*$ is negative at the point $(0, 1)$ (here, 0 is an r-dimensional vector). Thus,

$$V(x, y, p, z) = \frac{1}{2}\sum_{i=1}^{s}(x^i)^2 + w(z)^k + \sum_{i=1}^{m} f_i(y, p, z)\, y^i + \sum_{i,j=1}^{r} f_2(y, p, z)\, p^i p^j$$

There are no terms here of the type $p^i\,(z)^{k-1}$.

On the basis of the zero approximation, we find the vector function $\xi_1(t)$, which, when substituted into (2.2), causes all the terms of order $-\mu-2$ and below in t to vanish. We assert that this solution exists and has the form

$$\xi_1(t) = \mathrm{col}\,(x_1 t^{-\mu-2},\ 0,\ 0,\ z_1 t^{-\mu}),\quad \xi_1 = \mathrm{col}\,(x_1, 0, 0, z_1)$$

For, on substituting the vector $\xi_1(t)$ into Eq.(2.2) and neglecting all the terms of higher degrees, we find, on equating coefficients of $t^{-\mu-2}$

$$\begin{gathered} 0 = (aa^T - E_s)\,x_1 + ab^T V_y \\ 0 = ba^T x_1 + (bb^T - E_m)\,V_y,\quad V_y = \frac{\partial V}{\partial y}(\xi_1) \\ 0 = 0,\quad z_1\mu\,(\mu+1) = -\,wk\,(z_1)^{k-1} \end{gathered} \tag{3.2}$$

The last $r+1$ equations are the same as system (3.1), i.e., the latter can be satisfied by taking $z_1 = z_0$. In order to find V_y, we do not need to know x_1, since the separation of the variables into "quadratic" and others has already been made. Thus the first s equations are linear inhomogeneous in the vector x_1, while the matrix $aa^T - E_s$ is non-degenerate by Assertion 1. Consequently, it follows that the vector x_1 is uniquely defined. This vector also satisfies Eqs.(3.2) in projections onto the coordinate y-axes (i.e., the equations with numbers from $s+1$ to $s+m$) by virtue of the linear dependence of the rows of matrix T, see Assertion 2. The terms Γ and OV' that appear in (2.2) have degree not lower than $-2\mu-2$, so that the $\xi_1(t)$ obtained is the required first approximation.

4. ***Structure of the linear operator.*** We propose to seek the formal solution as

$$\begin{gathered} x^i(t) = \frac{1}{t^2}\sum_{j=1}^{\infty} x_j^i t^{-j\mu},\quad i = 1, 2, \ldots, s \\ y^i(t) = \sum_{j=2}^{\infty} y_j^i t^{-j\mu},\quad i = 1, 2, \ldots, m \\ p^i(t) = \sum_{j=2}^{\infty} p_j^i t^{-j\mu},\quad i = 1, 2, \ldots, r \\ z(t) = \sum_{j=1}^{\infty} z_j t^{-j\mu} \end{gathered} \tag{4.1}$$

Here, x_j^i, y^i_j, p_j^i, z_j are polynomials of the argument $\tau = \ln t$, when $j = 1$ we obtain the above first approximation $\xi_1(t)$. If, in spite of our assumption, $L_A \cap L_X = L_Q \neq \{0\}_x$ then the corresponding coordinates can be found as

$$q^i(t) = \frac{1}{t^2}\sum_{j=1}^{\infty} q_j^i t^{-\mu-j\mu}$$

The solution is formal, i.e., the convergence of the series is not considered.

Assume that the $(N-1)$-th approximation has been found (i.e., the series of type (4.1), where the summation is not to infinity but only to $N-1$), and that it satisfies Eqs.(2.2) up to and including terms of order $-(N-1)\mu-2$. We construct the N-th approximation, which satisfies (2.2) up to terms of order $-N\mu-2$ in t. Collecting terms of $t^{-N\mu-2}$, we obtain

$$\left\| \begin{matrix} 0 \\ y_N'' - (2N\mu+1)\,y_N' \\ p_N'' - (2N\mu+1)\,p_N' \\ z_N'' - (2N\mu+1)\,z_N' \end{matrix} \right\| - B_N \left\| \begin{matrix} x_N \\ y_N \\ p_N \\ z_N \end{matrix} \right\| = \eta \tag{4.2}$$

η is a vector polynomial of τ, whose coefficients are expressible in terms of the ξ_j obtained earlier. The prime denotes differentiation with respect to τ, and B_N is a constant linear operator. We have to solve system (4.2) (i.e., find the polynomials x_N, y_N, p_N, z_N)

for every N with any right-hand side. This will be done in the remainder of the proof.

5. ***Study of the linear operator.*** The matrix B_N is given by

$$B_N = TV''(\xi_1) - c_N E_{ypz}$$

where V'' is an $n \times n$ matrix composed of the second derivatives of the potential V, taken at the point ξ_1, $c_N = N\mu(N\mu + 1)$, E_{ypz} is a diagonal matrix whose first s diagonal elements are zeros, and the remaining $m + r + 1$ are unities. The block structure of B_N is

$$B_N = \left\| \begin{matrix} aa^T - E_s & ab^T V_{yy} & ab^T V_{yp} & ab^T V_{yz} \\ ba^T & (bb^T - E_m) V_{yy} - c_N E_m & (bb^T - E_m) V_{yp} & (bb^T - E_m) V_{yz} \\ 0 & -V_{py} & -V_{pp} - c_N E_r & 0 \\ 0 & -V_{zy} & 0 & -V_{zz} - c_N \end{matrix} \right\|$$

All the partial derivatives are taken at the point ξ_1.

Assertion 3. We have $\det B_N \neq 0$ for $N \neq N^0 = k/2$.

On subtracting from the y-rows the x-rows with the corresponding coefficients, we can arrange for the diagonal elements of the y-rows to be equal to $-c_N$, while the remaining elements of the rows are zero. Then, expanding the determinant with respect to them, we obtain

$$\det B_N = \text{const}\,(c_N)^m \det(aa^T - E_s) \det \left\| \begin{matrix} -V_{pp} - c_N E_r & 0 \\ 0 & -V_{zz} - c_N \end{matrix} \right\|$$

It has been said that $\det(aa^T - E_s) \neq 0$, while the second determinant likewise does not vanish for $N \neq N^0$, and with $N = N^0$ zero is a single eigenvalue of the matrix B_N, and $c_N = -V_{zz}$.

Assertion 4. The kernel of the matrix B_{N^0} is one-dimensional.

Let $\xi \in \operatorname{Ker} B_{N^0}$; then

$$\begin{gathered} (aa^T - E_s)x + ab^T(V_{yy}y + V_{yp}p + V_{yz}z) = 0 \\ ba^T x + (bb^T - E_m)(V_{yy}y + V_{yp}p + V_{yz}z) = c_{N^0} y \\ -V_{py}y - V_{pp}p - c_{N^0}p = 0, \quad -V_{zy}y = 0 \end{gathered}$$

Considering the previously used linear combination of the first s-rows, we obtain $y = 0$ and $p = 0$; but then, given z, the vector x is uniquely restored. The kernel of B_{N^0} is thus one-dimensional and is generated by the vector $\xi_* = (x_*, 0, 0, 1)^T$.

Assertion 5. The vector $(0, 0, 0, 1)^T$ does not belong to $\operatorname{Im} B_{N^0}$.

The proof is by reductio ad absurdum, using the block structure of B_{N^0}.

6. ***Proof that (4.2) is solvable.*** For every N, the solution must be a vector polynomial of τ. We shall construct it by starting with the highest powers of τ that appear on the right-hand side, and reducing this power each time by unity. It is assumed that the highest remaining power is the j-th, and that its coefficient is $\alpha = (\alpha^1, \ldots, \alpha^n)^T$.

Case 1. $N \neq N^0$. The linear operator B_N is then not degenerate, and we take as the coefficient of $(\tau)^j$ in ξ_N the vector $-B_N^{-1}\alpha$.

Case 2. $N = N^0$. We do not exclude that $\alpha \notin \operatorname{Im} B_{N^0}$; we resolve α into two components: $\alpha = \beta + \gamma$, where $\beta \in \operatorname{Im} B_{N^0}$, and $\gamma = (0, \ldots, 0, \gamma^n)$. This is possible, by Assertion 5. We then take the corresponding components in ξ_{N^0} equal to

$$-B_{N^0}^{-1}\beta(\tau)^j - \gamma^n \xi_*(\tau)^{j+1}/[(j+1)(2N\mu + 1)]$$

On substituting into (4.2) we obtain the required $\alpha(\tau)^j$.

We have thus shown that (4.2) is solvable, so that the formal solution of system (1.2) is obtained. The proof of the lemma is complete.

In view of what we said above, the theorem is also proved.

7. ***Some corollaries.*** *Corollary 1.* If the conditions of the theorem hold, system (1.2) has asymptotic motions. The number of these motions is not less than the number of local minimum points of the function $V_k{}^*$ in the unit sphere of L_B, at which it is negative.

Corollary 2. If we dispense with the non-holonomic constraints $(m = 0)$, we obtain Theorem 1 of /6/, where a similar criterion is obtained for natural mechanical systems.

Corollary 3. It is clear from the proof that, when k is odd, the formal solution contains no logarithms.

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AN ASYMPTOTIC ANALYSIS OF THE FORCED OSCILLATIONS IN SYSTEMS WITH SLOWLY VARYING PARAMETERS*

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The oscillations in weakly non-linear systems with slowly varying parameters are investigated. For periodically varying parameters, a spectral analysis is made of the steady-state oscillations in order to obtain reasonably simple analytical results. Special attention is paid to the cases when some natural frequencies vary over a much wider range than the frequency of parameter variation.

The usual basic methods for analysing such problems /1-3/ are not suitable of the present purpose, especially when the parameters vary over a wide range. A rather different scheme for analysing the system of differential equations is proposed below. The matrizant (Green's function) of the linear problem is written in a form which ensures faster convergence than in the WKB method and of the procedure for the asymptotic evaluation of the required quantities /1, 2/. Even to a first approximation, the results differ from those of /1, 2/, and differ the more, the greater the range of variation of the parameters. The non-linear forces are taken into account by successive approximation

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